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Paths of Spinning Particles in General Relativity as Geodesics of an Einstein Connection

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Abstract

This paper deals with the Papapetrou–Pirani equations of motion for a spinning test particle in general relativity. The motion of the center of mass can be represented by the geodesic equation of an affine connection that is the sum of the Christoffel connection and a tensor that depends on the Riemann–Christoffel curvature tensor, the mass of the particle, its 4-velocity, and its spin tensor. The connection is not unique, and here it is chosen to satisfy one of the basic geometrical principles of Einstein's unified field theory: The symmetric part of the fundamental tensor of the geometry is specified to be the metric tensor of general relativity. The special case of conformally flat space-times is discussed.

1. *Introduction*

In view of the difficulties that have been experienced in attempts to obtain unified field theories, it appears to be worthwhile pursuing the less ambitious approach of attempting to write known equations in a more unified form. In particular, one of the major problems with Einstein's unified field theory has been that of establishing the physical interpretations of the mathematical quantities involved; expressing equations of general relativity in terms of Einstein's non-Riemannian geometry might provide information as to what interpretations are possible.

The generally covariant equation of motion of a test charge in an external electromagnetic field, with radiation reaction neglected, has been written as the equations of geodesics in four-dimensional Finsler space, in five-dimensional Riemannian space (Sen, 1968, p. 85), and in a six-dimensional non-Riemannian space (Bown, 1970). It has also been expressed as the geodesic equation of an affine connection within a four-dimensional space-time having the usual

Riemannian metric (Droz-Vinçent, 1967); the connection is the sum of the Christoffel connection and a third-rank tensor that depends on the electromagnetic field and on the 4-velocity and charge-to-mass ratio of the particle. The connection is not unique, there being a class of connections having the same geodesics; it can be chosen so that the covariant derivative of the metric tensor with respect to the connection vanishes (Droz-Vincent, 1967), which property is preserved if any further skew part giving no contribution to the torsion vector is added to it. The connection can instead be chosen (Burman, 1971a, 1971b) to be an Einstein connection—a connection satisfying one of the basic geometrical principles of the nonsymmetric unified field theory developed by Einstein, Schrödinger, and others, namely, the condition relating the connection to the nonsymmetric fundamental tensor. An alternative condition can be imposed (Burman, 1971a), namely, the vanishing of the covariant derivative, with respect to the connection, of the nonsymmetric fundamental tensor.

A considerable amount of work has been done on the relation between spin and torsion (e.g., Hehl, 1973), In particular, Sciama (1958a, 1958b, 1961, 1962, 1964) has suggested that Einstein's unified theory is a unification of gravitation and spin. In view of this, it is of interest to express the motion of a spinning particle in general relativity as the geodesic equation of an Einstein connection.

2. *Basic Theory*

In general relativity matter can be described by an energy-momentum tensor or by singularities in the field with the empty-space field equations applicable outside the singularities (Bondi, 1959). With the former description it is easily shown that the equation of motion of incoherent matter follows from the field equations; in the absence of nongravitational fields the particles follow geodesics of the Riemannian space of general relativity; in particular, this result applies to a single test particle, as is seen by taking the density to be proportional to a delta function (Sen, 1961; 1968 p. 20). Einstein, Infeld, and Hoffman derived the equations of motion of gravitating particles in their total gravitational field by using the second description (Infeld and Plebański, 1960); they dealt with bodies having comparable masses and moving slowly compared with light. A different method, introduced by Fock (1964, ch. 6) and developed by Papapetrou (1951a), brings the interiors of the bodies into account; it uses the field equations and the condition that the covariant divergence of the energy-momentum tensor of a body vanishes. On being applied to the simplest kind of test particle, both methods show the path to be a geodesic.

The second method was used by Papapetrou (1951b) to investigate the motion of a spinning test particle in the absence of nongravitational fields; the particle is an elementary torque-free gyroscope. Equations of motion for its center of mass and for its internal angular momentum or spin tensor ($S^{\alpha\beta}$) were obtained and expressed in covariant form. These equations are not sufficient to determine all unknowns: There are three equations of motion for the six independent components of the skew-symmetric spin tensor. Some conditions must be imposed on the $S^{\alpha\beta}$, and Pirani (1956) suggested taking

$$u_\beta S^{\alpha\beta} = 0 \tag{2.1}$$

which will be adopted here.

Consider the four-dimensional Riemannian space-time of general relativity with x^α the coordinates and ds the interval: $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ where $(g_{\mu\nu})$ is the metric tensor. The speed of light in empty space will be put equal to unity. A particle with mass m is at (x^α) and has 4-velocity $(u^\mu) \equiv (dx^\mu/ds)$. With (2.1), Papapetrou's equations combine to show that m is a constant of the motion and to give

$$m\dot{u}^\mu = -(\ddot{S}^\mu{}_\nu + \tfrac{1}{2}R^\mu{}_{\nu\rho\sigma}S^{\rho\sigma})u^\nu \tag{2.2}$$

where a dot denotes total covariant differentiation following the world-line of the particle and $(R^\alpha{}_{\beta\mu\nu})$ is the Riemann-Christoffel curvature tensor. Let a comma denote partial differentiation and a semicolon denote covariant differentiation with respect to the Christoffel connection. Since $\dot{u}^\mu = u^\mu{}_{;\nu}u^\nu$ and

$$u^\mu{}_{;\nu} = u^\mu{}_{,\nu} + \begin{Bmatrix} \mu \\ \alpha\nu \end{Bmatrix} u^\alpha \tag{2.3}$$

where the braces denote the Christoffel symbol of the second kind, (2.2) can be expressed in the alternative form

$$\frac{d^2x^\mu}{ds^2} + \begin{Bmatrix} \mu \\ \alpha\beta \end{Bmatrix} \frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = -\frac{1}{m}\left(\ddot{S}^\mu{}_\nu + \frac{1}{2}R^\mu{}_{\nu\rho\sigma}S^{\rho\sigma}\right)u^\nu \tag{2.4}$$

3. *The Affine Connection*

Consider an entity Δ with components defined by

$$\Delta_\alpha{}^\mu{}_\beta \equiv \begin{Bmatrix} \mu \\ \alpha\beta \end{Bmatrix} + A_\alpha{}^\mu{}_\beta \tag{3.1}$$

where $(A_\alpha{}^\mu{}_\beta)$ is a third-rank tensor. The sum of a connection and a tensor of the appropriate type is always a connection, so Δ is one. Covariant differentiation with respect to Δ will be denoted by a stop. Hence

$$u^\nu u^\mu{}_{.\nu} = u^\nu(u^\mu{}_{;\nu} + A_\alpha{}^\mu{}_\nu u^\alpha) \tag{3.2}$$

Substituting (2.2) into (3.2), it is seen that if $(A_\alpha{}^\mu{}_\beta)$ satisfies

$$A_\alpha{}^\mu{}_\beta u^\alpha u^\beta = \frac{1}{m}\left(\ddot{S}^\mu{}_\alpha + \frac{1}{2}R^\mu{}_{\alpha\rho\sigma}S^{\rho\sigma}\right)u^\alpha \tag{3.3}$$

then

$$u^\nu u^\mu{}_{.\nu} = 0 \tag{3.4}$$

Equation (3.4) can be written in the alternative form

$$\frac{d^2x^\mu}{ds^2} + \Delta_\alpha{}^\mu{}_\beta \frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0 \tag{3.5}$$

which can be obtained directly by using (3.1) and (3.3) to eliminate the Christoffel symbol from (2.4). Equation (3.4) is the equation of the geodesics of the affine connection Δ: The paths of spinning test particles can be described by such geodesics.

From the general expression for a covariant derivative it follows that

$$g_{\mu\nu.\rho} = g_{\mu\nu;\rho} - A_{\mu\nu\rho} - A_{\nu\mu\rho} \tag{3.6}$$

Hence if $A_{\alpha\beta\gamma} = -A_{\beta\alpha\gamma}$, then

$$g_{\mu\nu.\rho} = 0 \tag{3.7}$$

Equation (3.3) is satisfied if

$$A_\alpha{}^\mu{}_\beta u^\beta = -\frac{1}{m}\left(\ddot{S}_\alpha{}^\mu + \frac{1}{2}R_\alpha{}^\mu{}_{\rho\sigma}S^{\rho\sigma}\right) \tag{3.8}$$

where use has been made of the skew-symmetry of the spin tensor and the skew-symmetry of the Riemann–Christoffel tensor in its first two indices. Now choose $(A_\alpha{}^\mu{}_\beta)$ to have the form given by

$$A_\alpha{}^\mu{}_\beta = B_\alpha{}^\mu u_\beta \tag{3.9}$$

with $(B_{\alpha\mu})$ a skew-symmetric second rank tensor; (3.7) is satisfied. Since $u_\nu u^\nu = 1$, the condition (3.8) becomes

$$B_\alpha{}^\mu = -\frac{1}{m}\left(\ddot{S}_\alpha{}^\mu + \frac{1}{2}R_\alpha{}^\mu{}_{\rho\sigma}S^{\rho\sigma}\right) \tag{3.10}$$

thus $(B_{\alpha\mu})$ is skew, as required.

That is, the equation of motion of the center of mass of a spinning test particle in general relativity can be expressed in the form of the geodesic equation of an affine connection Δ with components given by

$$\Delta_\alpha{}^\mu{}_\beta = \left\{ {\mu \atop \alpha\beta} \right\} - \frac{1}{m}\left(\ddot{S}_\alpha{}^\mu + \frac{1}{2}R_\alpha{}^\mu{}_{\rho\sigma}S^{\rho\sigma}\right) u_\beta \tag{3.11}$$

This connection satisfies the condition $g_{\mu\nu.\rho} = 0$. The torsion vector of Δ is given by

$$\Delta_\lambda \equiv \Delta_{[\lambda}{}^\sigma{}_{\sigma]} \tag{3.12a}$$

$$= \frac{-1}{2m}\left(\ddot{S}_{\lambda\tau} + \frac{1}{2}R_{\lambda\tau\rho\sigma}S^{\rho\sigma}\right)u^{\tau} \tag{3.12b}$$

Comparison of (3.12b) with (2.2) shows that the torsion vector is proportional to the effective 4-force acting on the particle. Proportionality of the 4-force with the torsion vector of the relevant connection has also been found for certain other equations of motion when the corresponding affine connections are required to satisfy $g_{\mu\nu.\rho} = 0$, namely, that for a test particle acted on by a field with an energy-momentum tensor $(T_{\mu\nu})$ satisfying $u_\mu T^{\mu\nu}{}_{;\nu} = 0$ (Burman, 1970), which includes the case of a test charge in an electromagnetic field, and that for a charged particle when electromagnetic radiation reaction is included (Burman, 1971c).

If (V_α) is a vector and $(C_\alpha{}^\mu{}_\beta)$ is a tensor that is skew in its first and last indices, then the quantities

$$D_\alpha{}^\mu{}_\beta = \Delta_\alpha{}^\mu{}_\beta + 2\delta^\mu_{(\alpha} V_{\beta)} + C_\alpha{}^\mu{}_\beta \tag{3.13}$$

form a connection D with the same geodesics as Δ (Schrödinger, 1950, p. 55). Let a colon denote covariant differentiation with respect to D; in particular,

$$g_{\mu\nu:\rho} = g_{\mu\nu,\rho} - g_{\alpha\nu}D_\mu{}^\alpha{}_\rho - g_{\mu\alpha}D_\nu{}^\alpha{}_\rho \tag{3.14a}$$

$$= g_{\mu\nu.\rho} - 2g_{\mu\nu}V_\rho - 2g_{\rho(\mu}V_{\nu)} - 2C_{(\mu\nu)\rho} \tag{3.14b}$$

Consider the vector (C_α) given by

$$C_\alpha \equiv C_\alpha{}^\sigma{}_\sigma = -C_\sigma{}^\sigma{}_\alpha \tag{3.15}$$

If $(C_{\alpha\mu\beta})$ is skew-symmetric in its first two indices, then (C_α) vanishes; if in addition (V_α) vanishes, then from (3.14b) $g_{\mu\nu:\rho} = g_{\mu\nu.\rho}$. The geodesics of a connection are unaffected by its skew part. The connection Δ defined by (3.11) is nonsymmetric, as it must be for the condition $g_{\mu\nu.\rho} = 0$ to hold: The tensor $(C_\alpha{}^\mu{}_\beta)$ that would cancel with the skew part of Δ has a nonzero associated vector (C_α), equal to $-(\Delta_\alpha)$, and so (3.15) shows that $(C_{\alpha\mu\beta})$ would not be skew in its first two indices; it will be shown later that such an addition is not admissible.

A preliminary note covering some of the work presented in this section has appeared previously (Burman, 1971d).

4. *The Einstein Connection*

4.1. *Einsteinian Geometry.* Let $(g_{\mu\nu})$ now denote a nonsymmetric tensor field, called the fundamental tensor, and write $h_{\mu\nu}$ for $g_{(\mu\nu)}$ and $k_{\mu\nu}$ for $g_{[\mu\nu]}$. Round and square brackets around indices will denote symmetric and skew parts taken with respect to the indices immediately inside the brackets. Let indices other than those of the fundamental tensor be raised and lowered by using $h_{\mu\nu}$ and $h^{\mu\nu}$, where the contravariant components $h^{\mu\nu}$ are defined by

$$h_{\lambda\mu}h^{\lambda\nu} = \delta_\mu{}^\nu \tag{4.1}$$

The contravariant components $g^{\mu\nu}$ of the fundamental tensor are defined by

$$g_{\lambda\mu}g^{\lambda\nu} = g_{\mu\lambda}g^{\mu\nu} = \delta_\lambda^{\ \nu} \tag{4.2}$$

Compare the conditions

$$g_{\mu\nu.\rho} = 0 \tag{4.3}$$

and

$$g_{\mu+\nu-.\rho} = 0 \tag{4.4}$$

where

$$g_{\mu\nu.\rho} \equiv g_{\mu\nu,\rho} - g_{\alpha\nu}\Gamma_{\mu\ \rho}^{\ \alpha} - g_{\mu\alpha}\Gamma_{\nu\ \rho}^{\ \alpha} \tag{4.5}$$

and

$$g_{\mu+\nu-.\rho} \equiv g_{\mu\nu,\rho} - g_{\alpha\nu}\Gamma_{\mu\ \rho}^{\ \alpha} - g_{\mu\alpha}\Gamma_{\rho\ \nu}^{\ \alpha} \tag{4.6}$$

in which Γ is a nonsymmetric affine connection. The definitions (4.5) and (4.6) differ only through the orders of the indices ν and ρ in their last terms. Equation (4.4), a basic geometrical principle of the unified field theory developed by Einstein and Schrödinger, has been studied in great detail by Hlavatý (1958); other authors have obtained solutions in various forms.

It has been shown (Hlavatý, 1958, p. 58) that the requirements for the existence and uniqueness of a solution of (4.4) do not impose any restriction in the form of an equation on the components $g_{\mu\nu}$. A necessary condition for (4.3) to have a solution is that (Hlavatý, 1958, p. 48) $\det(g_{\mu\nu})/\det(h_{\mu\nu})$ and $\det(k_{\mu\nu})/\det(h_{\mu\nu})$ must both be constant. Thus the condition (4.4) produces a more flexible theory than (4.3).

Consider a connection Γ with components written in the form

$$\Gamma_{\alpha\ \beta}^{\ \mu} = \left\{ \begin{matrix} \mu \\ \alpha\beta \end{matrix} \right\} + U^{\mu}{}_{\alpha\beta} + S_{\alpha\beta}{}^{\mu} \tag{4.7}$$

where the Christoffel symbols are defined in terms of $(h_{\mu\nu})$, $(S_{\alpha\beta}{}^{\mu})$ is the torsion tensor of Γ,

$$S_{\alpha\beta}{}^{\mu} \equiv \Gamma_{[\alpha\ \beta]}^{\ \mu} \tag{4.8}$$

and

$$U^{\mu}{}_{\alpha\beta} \equiv 2S^{\mu}{}_{(\alpha}{}^{\nu}k_{\beta)\nu} \tag{4.9}$$

It has been shown (Hlavatý, 1958, p. 52) that if (4.4) has a solution Γ, then it must be of the form described by (4.7), (4.8), and (4.9). Equation (4.9) implies that (Hlavatý, 1958, p. 62)

$$U_{\nu\lambda\mu} + U_{\mu\nu\lambda} + U_{\lambda\mu\nu} = 0 \tag{4.10}$$

—another theorem for use below.

4.2. *An Einstein Connection.* Since the geodesics of a connection are unaffected by a transformation of the form (3.13), it is of interest to attempt to find a connection D, related by such a transformation to Δ defined by (3.11), that will satisfy a condition of the type (4.4). The tensor $(h_{\mu\nu})$ is specified to be the Riemannian metric tensor of general relativity, so the Christoffel symbols in (3.11) and (4.7) are the same and $(R^{\alpha}{}_{\beta\mu\nu})$ denotes the Riemann-Christoffel curvature tensor of $(h_{\mu\nu})$. It is seen that D can be written in the form given by (4.7) with

$$U^{\mu}{}_{\alpha\beta} = B_{(\alpha}{}^{\mu}u_{\beta)} + 2\delta_{(\alpha}{}^{\mu}V_{\beta)} \tag{4.11}$$

where $B_{\alpha}{}^{\mu}$ is given by (3.10), and with some suitable skew-symmetric tensor chosen for $(S_{\alpha\beta}{}^{\mu})$. Taking

$$S_{\alpha\beta}{}^{\mu} = -\tfrac{1}{2}B_{\alpha\beta}L^{\mu} \tag{4.12}$$

where (L^{μ}) is some vector, (4.9), (4.11), and (4.12) give

$$B_{(\alpha\mu}(L^{\nu}k_{\beta)\nu} - u_{\beta)}) = 2h_{\mu(\alpha}V_{\beta)} \tag{4.13}$$

If V_{β} is put equal to zero, (4.13) is satisfied if

$$k_{\beta\nu}L^{\nu} = u_{\beta} \tag{4.14}$$

Taking $\det(k_{\mu\nu})$ to be nonzero, (4.14) implies that

$$L^{\mu} = \bar{k}^{\mu\sigma}u_{\sigma} \tag{4.15}$$

where $(\bar{k}^{\mu\nu})$ is defined by

$$\bar{k}^{\alpha\sigma}k_{\sigma\beta} = \delta_{\beta}{}^{\alpha} \tag{4.16}$$

With $V^{\mu} = 0$ and L^{μ} given by (4.15), equations (4.11) and (4.12) become

$$U^{\mu}{}_{\alpha\beta} = B_{(\alpha}{}^{\mu}u_{\beta)} \tag{4.17}$$

and

$$S_{\alpha\beta}{}^{\mu} = -\tfrac{1}{2}B_{\alpha\beta}\bar{k}^{\mu\sigma}u_{\sigma} \tag{4.18}$$

So

$$D_{\alpha}{}^{\mu}{}_{\beta} = \begin{Bmatrix} \mu \\ \alpha\beta \end{Bmatrix} + B_{(\alpha}{}^{\mu}u_{\beta)} - \tfrac{1}{2}B_{\alpha\beta}\bar{k}^{\mu\sigma}u_{\sigma} \tag{4.19}$$

This has the same symmetric part as Δ: Only the torsion is different, and that does not affect the geodesics. The torsion vector of D is given by

$$S_{\alpha} \equiv S_{\alpha\tau}{}^{\tau} \tag{4.20a}$$

$$= -\tfrac{1}{2}B_{\alpha\tau}\bar{k}^{\tau\sigma}u_{\sigma} \tag{4.20b}$$

Using (3.10), equations (4.17), (4.18), (4.19) and (4.20b) become

$$U^{\mu}{}_{\alpha\beta} = \frac{-1}{m}\,(\ddot{S}_{(\alpha}{}^{\mu} + \tfrac{1}{2}R_{(\alpha}{}^{\mu}{}_{\rho\sigma}S^{\rho\sigma})u_{\beta)} \tag{4.21}$$

$$S_{\alpha\beta}{}^{\mu} = \frac{1}{2m}(\ddot{S}_{\alpha\beta} + \tfrac{1}{2}R_{\alpha\beta\rho\sigma}S^{\rho\sigma})\bar{k}^{\mu\tau}u_{\tau} \tag{4.22}$$

$$D_{\alpha}{}^{\mu}{}_{\beta} = \begin{Bmatrix}\mu\\ \alpha\beta\end{Bmatrix} - \frac{1}{m}\,(\ddot{S}_{(\alpha}{}^{\mu} + \tfrac{1}{2}R_{(\alpha}{}^{\mu}{}_{\rho\sigma}S^{\rho\sigma})u_{\beta)} + \frac{1}{2m}\,(\ddot{S}_{\alpha\beta} + \tfrac{1}{2}R_{\alpha\beta\rho\sigma}S^{\rho\sigma})\bar{k}^{\mu\tau}u_{\tau} \tag{4.23}$$

and

$$S_{\alpha} = \frac{1}{2m}\,(\ddot{S}_{\alpha\tau} + \tfrac{1}{2}R_{\alpha\tau\rho\sigma}S^{\rho\sigma})\bar{k}^{\tau\beta}u_{\beta} \tag{4.24}$$

4.3. *Uniqueness.* Any tensor with components of the form $2\delta^{\mu}_{(\alpha}X_{\beta)} + T_{\alpha\beta}{}^{\mu}$, where (X_{α}) is a vector and the tensor $(T_{\alpha\beta\mu})$ is skew in its first two indices, can be added to D without affecting its geodesics. But for the resulting affinity to satisfy (4.4), the added terms must satisfy an equation of the form of (4.9) and hence an equation like (4.10). Substituting $2\delta^{\mu}_{(\alpha}X_{\beta)}$ for $U^{\mu}{}_{\alpha\beta}$ in (4.10) leads to

$$h_{\lambda\nu}X_{\mu} + h_{\mu\nu}X_{\lambda} + h_{\lambda\mu}X_{\nu} = 0$$

Raising ν and then contracting over λ and ν shows that X_{μ} must vanish. Hence (4.9) implies that

$$T_{\mu(\alpha}{}^{\nu}k_{\beta)\nu} = 0 \tag{4.25}$$

So the affinity D given by (4.23) is arbitrary up to the addition of a tensor $(T_{\alpha\beta}{}^{\mu})$ which is skew in its first two indices and satisfies (4.25). The condition (4.25) has arisen previously, in the problem of the representation of the motion of charged particles in electromagnetic fields as geodesics of an Einstein connection (Burman, 1971b).

Let (Hlavatý, 1958)

$$k \equiv \frac{\det(k_{\mu\nu})}{\det(h_{\mu\nu})} \tag{4.26a}$$

$$K \equiv \tfrac{1}{4}k_{\alpha\beta}k^{\alpha\beta} \tag{4.26b}$$

$$D \equiv K^2 - k \tag{4.26c}$$

$$^{(2)}k_{\lambda}{}^{\nu} \equiv k_{\lambda\alpha}k^{\alpha\nu} \tag{4.26d}$$

and

$$'k_{\omega\mu} = \tfrac{1}{2}\gamma\left(\frac{\det(h_{\alpha\beta})}{k}\right)^{1/2} e_{\omega\mu\lambda\nu}k^{\lambda\nu} \tag{4.26e}$$

where γ is the sign of a certain determinant and the indicator $(e_{\omega\mu\lambda\nu})$ is a totally skew tensor density of weight -1 having the components $1, -1$, or 0 according as the indices form an even permutation, an odd permutation, or no permutation of the numbers 1, 2, 3, and 4. Following Hlavatý (1958) take $(g_{\mu\nu})$ to be real and its determinant to be negative; $(h_{\mu\nu})$ has the signature $(+++-)$, so $\det(h_{\mu\nu}) < 0$; $(h_{\mu\nu})$ is the metric tensor; $(k_{\mu\nu})$ is not zero. Under these specifications, the condition (4.25) has been investigated (Burman, 1972) for the case in which $k \neq 0$. It was found that $T_{\lambda\mu}{}^{\nu}$ can be expressed in the form

$$T_{\lambda\mu}{}^{\nu} = M_{\lambda\mu}{}^{\nu\sigma} T_{\sigma} \tag{4.27}$$

where

$$\begin{aligned} -2DM_{\lambda\mu}{}^{\nu\sigma} &= k\delta^{\sigma}_{[\lambda}\delta^{\nu}_{\mu]} + {}^{(2)}k_{[\lambda}{}^{\sigma(2)}k_{\mu]}{}^{\nu} \\ &+ K(\delta^{\sigma}_{[\lambda}{}^{(2)}k_{\mu]}{}^{\nu} + {}^{(2)}k_{[\lambda}{}^{\sigma}\delta^{\nu}_{\mu]}) + \tfrac{1}{2}K(k'k_{\lambda\mu}{}'k^{\nu\sigma} + k_{\lambda\mu}k^{\nu\sigma}) \\ &+ \tfrac{1}{2}(K^2 + D)k_{\lambda\mu}{}'k^{\nu\sigma} + \tfrac{1}{2}(K^2 - D)'k_{\lambda\mu}k^{\nu\sigma} \end{aligned} \tag{4.28}$$

and

$$T_{\lambda} \equiv T_{\lambda\sigma}{}^{\sigma} \tag{4.29}$$

Thus the Einstein connection representing the equation of motion (2.2) is nonunique to the extent of the addition of a third-rank tensor $(T_{\lambda\mu\nu})$, skew in its first two indices, which is specified in terms of an arbitrary vector by (4.27) with (4.28).

In Einstein's unified field theory, one of the field equations is the requirement that the torsion vector of the connection must vanish. If this condition is imposed on the Einstein connection representing the equation of motion (2.2), then under the conditions specified above that connection is unique: T_{α} must equal $-S_{\alpha}$, so that, using (4.24),

$$T_{\alpha\beta}{}^{\mu} = -M_{\alpha\beta}{}^{\mu\sigma} S_{\sigma} \tag{4.30a}$$

$$= \frac{-1}{2m} M_{\alpha\beta}{}^{\mu\sigma}(\ddot{S}_{\sigma\tau} + \tfrac{1}{2}R_{\sigma\tau\gamma\delta}S^{\gamma\delta})\bar{k}^{\tau\epsilon}u_{\epsilon} \tag{4.30b}$$

4.4. *The Skew Part of the Fundamental Tensor.* The symmetric part of the fundamental tensor has been taken to be the metric tensor, but little has been said about $(k_{\mu\nu})$. In this subsection some equations that $(k_{\mu\nu})$ must satisfy will be derived.

A number of theorems will now be stated: First (Hlavatý, 1958, Theorem 2.3)

$$2S_{\omega\mu\nu} = K_{\omega\mu\nu} - 4U_{\alpha\nu[\mu}k_{\omega]}{}^{\alpha} \tag{4.31}$$

where

$$K_{\omega\mu\nu} \equiv g_{\nu\mu;\omega} + g_{\omega\nu;\mu} + g_{\omega\mu;\nu} \tag{4.32}$$

Second (Burman, 1971a)

$$k_{\mu\nu;\rho} + 2k_{[\nu\alpha}U^{\alpha}{}_{\mu]\rho} = 2S_{\rho[\nu\mu]} \tag{4.33}$$

Third (Hlavatý, 1958, p. 61),

$$k_{\mu\nu,\rho} + k_{\nu\rho,\mu} + k_{\rho\mu,\nu} = -2(S_{\mu\nu\rho} + S_{\nu\rho\mu} + S_{\rho\mu\nu}) \tag{4.34}$$

Fourth (Hlavatý, 1958, p. 67)

$$S_\lambda = k_\lambda{}^\alpha{}_{;\alpha} + 2U_\alpha k_\lambda{}^\alpha - U^\alpha{}_{\beta\lambda}k_\alpha{}^\beta \tag{4.35}$$

where $U_\mu \equiv U^\sigma{}_{\sigma\mu}$.

Replace (4.18) by

$$S_{\alpha\beta}{}^\mu = -\tfrac{1}{2}B_{\alpha\beta}\bar{k}^{\mu\sigma}u_\sigma + T_{\alpha\beta}{}^\mu \tag{4.36}$$

where $(T_{\alpha\beta}{}^\mu)$ has been defined above. Equations (4.17) and (4.36) give

$$U_\lambda = \tfrac{1}{2}B_{\lambda\sigma}u^\sigma \tag{4.37}$$

and

$$S_\lambda = -\tfrac{1}{2}B_{\lambda\tau}\bar{k}^{\tau\sigma}u_\sigma + T_\lambda \tag{4.38}$$

respectively; note that (U_λ) is proportional to the effective 4-force acting on the particle. Substituting (4.17) and (4.36) into first (4.31) and then (4.33) results in

$$k_{\nu\mu;\omega} + k_{\omega\nu;\mu} + k_{\omega\mu;\nu} = 2(B_{\nu\alpha}u_{[\mu} + B_{[\mu\alpha}u_\nu)k_{\omega]}{}^\alpha - B_{\omega\mu}\bar{k}_\nu{}^\sigma u_\sigma + 2T_{\omega\mu\nu} \tag{4.39}$$

and

$$k_{\mu\nu;\rho} + k_{[\nu\alpha}(B_{\mu]}{}^\alpha u_\rho + B_\rho{}^\alpha u_{\mu]}) + B_{\rho[\nu}\bar{k}_{\mu]}{}^\sigma u_\sigma = 2T_{\rho[\nu\mu]} \tag{4.40}$$

respectively. Substituting (4.36) into (4.34) gives

$$\begin{aligned} k_{\mu\nu,\rho} + k_{\rho\mu,\nu} + k_{\nu\rho,\mu} &= (B_{\mu\nu}\bar{k}_\rho{}^\sigma + B_{\rho\mu}\bar{k}_\nu{}^\sigma + B_{\nu\rho}\bar{k}_\mu{}^\sigma)u_\sigma \\ &\quad - 2(T_{\mu\nu\rho} + T_{\rho\mu\nu} + T_{\nu\rho\mu}) \end{aligned} \tag{4.41}$$

Substituting (4.17), (4.37), and (4.38) into (4.35) gives

$$k_\lambda{}^\alpha{}_{;\alpha} + B_\alpha{}^\sigma u_\sigma k_\lambda{}^\alpha - B_{(\lambda}{}^\alpha u_{\sigma)}k_\alpha{}^\sigma + \tfrac{1}{2}B_{\lambda\sigma}\bar{k}^{\sigma\tau}u_\tau = T_\lambda \tag{4.42}$$

Equations (4.39)-(4.42) are four partial differential equations relating the skew part of the fundamental tensor to the physical quantities m, (u_μ), $(S_{\mu\nu})$, and $(R^\mu{}_{\nu\rho\sigma})$.

If the torsion vector is required to vanish, then (4.42) reduces to

$$k_\lambda{}^\alpha{}_{;\alpha} + B_{\alpha\sigma}u^\sigma k_\lambda{}^\alpha - B_{(\lambda}{}^\alpha u_{\sigma)}k_\alpha{}^\sigma = 0 \tag{4.43}$$

5. *Conformally Flat Space-Times*

In a conformally flat space-time, $g_{\mu\nu}$ can be expressed as $e^\psi\eta_{\mu\nu}$ where $(\eta_{\mu\nu})$ is the metric tensor of flat space-time and ψ is a function of the x^μ. Also

(Petrov, 1969, Section 34)

$$R^{\alpha}{}_{\beta\gamma\delta} = \delta^{\alpha}_{[\gamma}(\tfrac{1}{2}\psi_{,\delta]}\,\psi_{,\beta} - \psi_{,\delta]\,,\beta} - \tfrac{1}{2}\eta_{\delta]\beta}\eta^{\sigma\tau}\psi_{,\sigma}\psi_{,\tau})$$
$$+ \eta^{\alpha\sigma}\eta_{\beta[\gamma}(\psi_{,\delta]\,,\sigma} - \tfrac{1}{2}\psi_{,\delta]}\,\psi_{,\sigma}) \qquad (5.1)$$

With (5.1), using the skew symmetry of $(S^{\mu\nu})$,

$$\tfrac{1}{2}R_{\alpha\beta\gamma\delta}S^{\gamma\delta} = (\psi_{,\sigma,[\alpha} - \tfrac{1}{2}\psi_{,\sigma}\psi_{,[\alpha})S_{\beta]}{}^{\sigma} - \tfrac{1}{4}\psi_{,\sigma}\psi^{,\sigma}S_{\alpha\beta} \qquad (5.2)$$

This gives, using the condition (2.1),

$$\tfrac{1}{2}R_{\alpha\beta\gamma\delta}S^{\gamma\delta}u^{\beta} = \tfrac{1}{2}(\tfrac{1}{2}\psi_{,\sigma}\psi_{,\beta} - \psi_{,\sigma,\beta})S_{\alpha}{}^{\sigma}u^{\beta} \qquad (5.3)$$

–hence (2.2) is specialized to conformally flat space-times. If

$$\psi_{,\alpha,\sigma} = \tfrac{1}{2}\psi_{,\alpha}\psi_{,\sigma} \qquad (5.4)$$

then the part of the effective 4-force that depends explicitly on the Riemann-Christoffel tensor vanishes.

The equation of motion (2.2) depends explicitly on the curvature tensor, and it is of interest to compare this dependence with that in another equation of motion that involves the curvature tensor. DeWitt and Brehme (1960) and Hobbs (1968a, b) investigated the motion of a nonspinning charged particle in an electromagnetic field in a Riemannian space of arbitrary hyperbolic metric, with the effect of electromagnetic radiation reaction included. The resulting equation of motion consists of the generally covariant form of the Lorentz–Dirac equation, together with extra force terms. The extra terms arise from fields that originate at the charge and are propagated back by the scattering effects of the space-time curvature: Huygen's principle fails in curved space-time. The force term found by DeWitt and Brehme is nonlocal in time, but does not depend explicitly on the curvature tensor; the term found by Hobbs contains the Ricci tensor. So the equation of motion contrasts with (2.2) which involves the full Riemann-Christoffel tensor. For the DeWitt-Brehme-Hobbs equation, the force term found by DeWitt and Brehme vanishes in conformally flat space-times; that found by Hobbs vanishes in conformally flat space-times satisfying (5.4). So in conformally flat space-times obeying (5.4), the curvature tensor, in its full or contracted forms, does not explicitly enter either the DeWitt–Brehme-Hobbs or Papapetrou–Pirani equations of motion: The equations have the same forms as in Minkowski space.

Conformally flat spaces that satisfy (5.4) have been discussed by Laugwitz (1965, Section 12.5). When (5.4) is satisfied, (5.1) becomes

$$R_{\alpha\beta\gamma\delta} = 2Kg_{\alpha[\gamma}g_{\delta]\beta} \qquad (5.5)$$

where

$$K = -\tfrac{1}{4}g^{\sigma\tau}\psi_{,\sigma}\psi_{,\tau} \qquad (5.6)$$

–the Riemann-Christoffel tensor has the form for a space-time of constant Riemannian curvature K; in general relativity, the space-time must therefore be de Sitter space (Synge, 1960, p. 256).

Using (5.2), equations (3.11) and (3.12b) become

$$\Delta_{\alpha\mu\beta} = h_{\mu\nu} \begin{Bmatrix} \nu \\ \alpha\beta \end{Bmatrix} - \frac{1}{m} [\ddot{S}_{\alpha\mu} + (\psi_{,\sigma,[\alpha} - \tfrac{1}{2}\psi_{,\sigma}\psi_{,[\alpha})S_{\mu]}{}^{\sigma} - \tfrac{1}{4}\psi_{,\sigma}\psi^{,\sigma}S_{\alpha\mu}]\, u_\beta \tag{5.7}$$

and, using the condition (2.1),

$$\Delta_\lambda = \frac{-1}{2m} [\ddot{S}_{\lambda\tau} + (\psi_{,\sigma,[\lambda} - \tfrac{1}{2}\psi_{,\sigma}\psi_{,[\lambda})S_{\tau]}{}^{\sigma}]u^\tau \tag{5.8}$$

If (5.4) is satisfied, then

$$\Delta_\alpha{}^\mu{}_\beta = \begin{Bmatrix} \mu \\ \alpha\beta \end{Bmatrix} - \frac{1}{m} (\ddot{S}_\alpha{}^\mu - \tfrac{1}{4}\psi_{,\sigma}\psi^{,\sigma}S_\alpha{}^\mu)u_\beta \tag{5.9}$$

and

$$\Delta_\lambda = \frac{-1}{2m} \ddot{S}_{\lambda\tau}u^\tau \tag{5.10}$$

Using (5.2), equations (4.23) and (4.24) become

$$\begin{aligned} D_\alpha{}^\mu{}_\beta = {} & \begin{Bmatrix} \mu \\ \alpha\beta \end{Bmatrix} - \frac{1}{m} [\ddot{S}_{(\alpha}{}^\mu + \tfrac{1}{2}(\psi_{,\sigma,(\alpha} - \tfrac{1}{2}\psi_{,\sigma}\psi_{,(\alpha})S^{\mu\sigma} \\ & - \tfrac{1}{2}(\psi_{,\sigma}{}^{,\mu} - \tfrac{1}{2}\psi_{,\sigma}\psi^{,\mu})S_{(\alpha}{}^{\sigma} - \tfrac{1}{4}\psi_{,\sigma}\psi^{,\sigma}S_{(\alpha}{}^\mu]u_{\beta)} \\ & + \frac{1}{2m} [\ddot{S}_{\alpha\beta} + (\psi_{,\sigma,[\alpha} - \tfrac{1}{2}\psi_{,\sigma}\psi_{,[\alpha})S_{\beta]}{}^{\sigma} - \tfrac{1}{4}\psi_{,\sigma}\psi^{,\sigma}S_{\alpha\beta}]\, \bar{k}^{\mu\tau}u_\tau \end{aligned} \tag{5.11}$$

and

$$S_\alpha = \frac{1}{2m} [\ddot{S}_{\alpha\tau} + (\psi_{,\sigma,[\alpha} - \tfrac{1}{2}\psi_{,\sigma}\psi_{,[\alpha})S_{\tau]}{}^{\sigma} - \tfrac{1}{4}\psi_{,\sigma}\psi^{,\sigma}S_{\alpha\tau}]\, \bar{k}^{\tau\beta}u_\beta \tag{5.12}$$

If (5.4) is satisfied, then

$$D_\alpha{}^\mu{}_\beta = \begin{Bmatrix} \mu \\ \alpha\beta \end{Bmatrix} - \frac{1}{m} (\ddot{S}_{(\alpha}{}^\mu - \tfrac{1}{4}\psi_{,\sigma}\psi^{,\sigma}S_{(\alpha}{}^\mu)u_{\beta)} + \frac{1}{2m} (\ddot{S}_{\alpha\beta} - \tfrac{1}{4}\psi_{,\sigma}\psi^{,\sigma}S_{\alpha\beta})\bar{k}^{\mu\tau}u_\tau \tag{5.13}$$

and

$$S_\alpha = \frac{1}{2m} (\ddot{S}_{\alpha\tau} - \tfrac{1}{4}\psi_{,\sigma}\psi^{,\sigma}S_{\alpha\tau})\bar{k}^{\tau\beta}u_\beta \tag{5.14}$$

Using (5.2), equation (4.30b), which is valid if the torsion vector of the Einstein connection is chosen to vanish, becomes

$$T_{\alpha\beta}{}^\mu = \frac{-1}{2m} M_{\alpha\beta}{}^{\mu\gamma} [\ddot{S}_{\gamma\delta} + (\psi_{,\sigma,[\gamma} - \tfrac{1}{2}\psi_{,\sigma}\psi_{,[\gamma})S_{\delta]}{}^{\sigma} - \tfrac{1}{4}\psi_{,\sigma}\psi^{,\sigma}S_{\gamma\delta}]\, \bar{k}^{\delta\epsilon}u_\epsilon \tag{5.15}$$

If (5.4) is satisfied, then (5.15) reduces to

$$T_{\alpha\beta}{}^{\mu} = \frac{-1}{2m} M_{\alpha\beta}{}^{\mu\gamma}(\ddot{S}_{\gamma\delta} - \tfrac{1}{4}\psi_{,\sigma}\psi^{,\sigma}S_{\gamma\delta})\bar{k}^{\delta\epsilon}u_{\epsilon} \tag{5.16}$$

6. *Concluding Remarks*

Hlavatý built his treatment of Einstein's unified field theory on three basic principles. Principle A asserts that the unified field of gravitation and electromagnetism is determined by 16 potentials, which are the components of a real second-rank nonsymmetric tensor field $(g_{\mu\nu})$; the symmetric and skew parts are denoted by $(h_{\mu\nu})$ and $(k_{\mu\nu})$. This principle relates to the algebraic structure imposed on the four-dimensional space-time by $(g_{\mu\nu})$: Hlavatý classified space-time, or either of the tensor fields $(g_{\mu\nu})$ and $(k_{\mu\nu})$, into first, second, and third classes, according as none, two, or all four of the eigenvalues of $(k_{\mu}{}^{\nu})$ vanish.

Principle B asserts that the potentials $g_{\mu\nu}$ determine the curvature and torsion of space-time: The tensor $(g_{\mu\nu})$ imposes a structure on the differential geometry of space-time through the connection Γ, which is defined in terms of $(g_{\mu\nu})$ by the set of 64 equations

$$g_{\underset{+}{\mu}\underset{-}{\nu};\rho} = 0 \tag{6.1}$$

For a given fundamental tensor $(g_{\mu\nu})$, the set (6.1) can have no solutions, a unique solution, or more than one solution. In four dimensions, with $(g_{\mu\nu})$ real and with $(h_{\mu\nu})$ having the signature (+++−), necessary and sufficient conditions for the existence and uniqueness of a solution of (6.1) are (Hlavatý, 1958) $g \neq 0$ and $g(g - 2) \neq 0$ for the first and second classes, respectively, where $g \equiv \det(g_{\mu\nu})/\det(h_{\mu\nu})$; for the third class, there is always a unique solution. Hlavatý solved (6.1) to obtain Γ in tensorial form, in terms of the $g_{\mu\nu}$ and their first covariant derivatives with respect to the Christoffel connection of $(h_{\mu\nu})$; he investigated all three classes including singular cases.

Principle C asserts that $(g_{\mu\nu})$ is a solution of a system of differential equations, the field equations, which impose conditions on the curvature and torsion of space-time. The procedure is to substitute a solution of (6.1) into the definition of the curvature tensor $(R_{\omega\mu\lambda}{}^{\nu})$ of Γ, thus obtaining $(R_{\omega\mu\lambda}{}^{\nu})$ in terms of the $g_{\mu\nu}$ and their first two derivatives. The tensor $(g_{\mu\nu})$ is then to be obtained from the field equations, as is the identification of the gravitational and electromagnetic fields in terms of the $g_{\mu\nu}$. Various sets of possible field equations have been proposed, including several sets that have in common the equations

$$\Gamma_{\alpha} = 0 \tag{6.2}$$

The motion of a spinning test particle in the Riemannian space-time of general relativity, with nongravitational fields neglected, can be described by the geodesic equation of an affine connection. The connection is the sum of the usual Christoffel connection and a tensor that depends on the Riemann-

Christoffel curvature tensor and on the particle's mass, 4-velocity, and spin tensor. In this paper, the connection has been chosen so as to satisfy the basic geometrical principle (6.1) of the Einstein-Schrödinger unified field theory. The symmetric part of the fundamental tensor has been put equal to the metric tensor of general relativity, and attention has been restricted to fields $(k_{\mu\nu})$ of the first class; $(k_{\mu\nu})$ has not been specified, but differential equations relating it to physical quantities have been found. The resulting Einstein connection has been found to be arbitrary to the extent that the torsion vector can be freely specified. In the unified field theory, the field equations (6.2) state that the torsion vector of the connection vanishes. If this condition is imposed on the Einstein connection obtained here, then that connection is unique.

The problem of satisfying further field equations, and that of identifying $(k_{\mu\nu})$in terms of physical quantities, remain to be investigated, as do the problems of dealing with fields $(k_{\mu\nu})$ of the second and third classes and singular cases and of removing the specification that the symmetric part of the fundamental tensor must be the metric tensor of general relativity.

The Einstein connection obtained in this paper is not an external property of space-time, independent of the test particle, but depends on the particle's mass, 4-velocity, and spin tensor. This may be physically interpreted to mean that the particle generates, on its own world-line, a modification of the space-time geometry. This interpretation has been suggested by Quale (1972) and Cohn (1972), who investigated the geodesic representation of the motion of a particle moving under electromagnetic and other forces. The dependence of the connection on the 4-velocity of the particle concerned suggests the problems of formulating the theory in phase space and Finsler space.

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